Online Appendix to "The Rise of Passive Investing and Index-linked Comovement"

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A Alternative measure of comovement

Chen, Singal, and Whitelaw (2016) (CSW) propose an alternative measure of comovement based on univariate regressions which is more appropriate in the case of event studies. The measurement issues raised by CSW are quite valid in the context of an event study in which a particular stock is switching index. In this section, I first discuss why I do not believe that this alternative measure is appropriate in my setup and then present results estimated using the CSW approach.

A.1 Discussion of the univariate regression approach

My setup is different than the one disucssed by CSW in that I am measuring time series changes in the average comovement among all index stocks. In this context, the univariate regression approach of CSW is not appropriate as it cannot capture the effect under study.

To illustrate this, let's assume the driving processes for returns are

$$
\begin{aligned}
y_t &= b_1 f_t + c_1 u_{1t} + e_{yt}, \\
x_{1t} &= b_1 f_t + c_1 u_{1t} + e_{1t}, \\
x_{2t} &= b_2 f_t + c_2 u_{2t} + e_{2t}, \qquad \text{(A.1)}
\end{aligned}
$$

where x_{1t} represents an index comprised of a large number of stocks with the same return process y_t and x_{2t} is a second index. Because all stocks in index 1 have the same loadings b_1 and c_1 on f_t and u_{1t}, index 1 also has the same loadings.

In this context, an increase in average index-linked comovement would come from an increase in c_1 and a corresponding decrease in b_1, assuming that the total variance of the index in unaffected.

In the univariate regressions

$$y_t = \alpha + \beta_1 x_{1t} + \varepsilon_t,$$
$$y_t = \alpha + \beta_2 x_{2t} + \varepsilon_t, \tag{A.2}$$

the probability limit of the slope coefficient estimates are

$$\beta_1 = \frac{cov(y_t, x_{1t})}{var(x_{1t})},$$
$$\beta_2 = \frac{cov(y_t, x_{2t})}{var(x_{2t})}. \tag{A.3}$$

With the given dynamics, the slope estimates become

$$\beta_1 = \frac{b_1^2\sigma_f^2 + c_1^2\sigma_{u1}^2}{b_1 b_2\sigma_f^2 + c_1^2\sigma_{u1}^2 + \sigma_{\varepsilon 1}^2},$$
$$\beta_2 = \frac{b_1^2\sigma_f^2}{b_2^2\sigma_f^2 + c_2^2\sigma_{u2}^2 + \sigma_{\varepsilon 2}^2}. \tag{A.4}$$

Clearly, an increase in c_1 would not cause any change in β_2, and β_1 would remain very close to one, assuming that $\sigma_{\varepsilon 1}^2$ is small relative to $var(x_{1t})$. If the increase in c_1 is accompanied by a decrease in b_1, then β_2 would decrease. This corresponds to empirical results obtained by replicating the main results of the paper using the univariate regression approach presented in the next section.

A.2 Main results revisited with univariate regressions

To match the univariate regression approach of CSW, I estimate comovement β^{univ}s from the following regressions:

$$
\begin{aligned}
R_{j,t} &= \alpha_{SP500,j} + \beta^{univ}_{j,\mathrm{SP500},t} R_{\mathrm{SP500},t} + u_{SP500,j,t} \\
R_{j,t} &= \alpha_{nonSP500,j} + \beta^{univ}_{j,\mathrm{nonSP500},t} R_{\mathrm{nonSP500},t} + u_{nonSP500,j,t},
\end{aligned}
\tag{A.5}
$$

where $R_{\mathrm{SP500},t}$ is the value-weighted return of the S&P 500 stocks portfolio (excluding stock j) and $R_{\text{non-SP500},t}$ is the value-weighted return of the rest of the market.

Figure A.1 of the appendix replicates Figure 6 from the main text, replacing the bivariate βs by the CSW univariate βs. Consistent witth the discussion in the previous section, I find that the average $\beta^{univ}_{\mathrm{SP500},t}$ is stable through time, remaining close to one. The average $\beta^{univ}_{\mathrm{nonSP500},t}$ is also quite stable close two one, with the exception of two outlier years.

I next reestimate the regressions presented in Table 1 of the main text using the CSW univariate βs. Results presented in Table A.1 of this appendix show that the estimate for $\Delta\beta_1$ is small and negative, while the estimate for $\Delta\beta_2$ is larger and negative. However, none of these estimates is statistically significant.

B Proofs and model derivation

B.1 Setup

This section provides additional details on the model presented in Section 3.

B.1.1 Information structure

Uncertainty is represented by a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathcal{P})$ on which is defined a 3-dimensional vector of independent Brownian motions $Z = [Z_1 \ Z_2 \ Z_3]'$. The filtration $\mathbf{F} = \{\mathcal{F}_t\}$ is the augmentation under $\mathcal{P}$ of the filtration generated by Z. The σ-field $\mathcal{F}_t$ represents the information available at time t and the probability measure $\mathcal{P}$ represents the agent's common beliefs. Stochastic processes to follow are progressively measurable with respect to $\mathbf{F}$ and equalities involving random variables hold $\mathcal{P}$-a.s.

B.1.2 Consumption space

There is a single perishable good, the numeraire. The agents' consumption set $\mathcal{C}$ is given by the set of non-negative progressively measurable consumption rate process c_t with $\int_0^T |c_t| dt < \infty, \ \forall T \in [0, \infty)$.

B.1.3 Securities market

The investment opportunities are represented by a locally riskless bond earning the instantaneous interest rate r and three risky stocks, representing claims to exogenously given strictly positive dividend processes $D_i, \ i = 1, 2, 3$, with

$$\frac{dD_{i,t}}{D_{i,t}} = \mu_D dt + \sigma_D dZ_{D_{i,t}}, \ i \in \{1, 2, 3\}, \tag{A.1}$$

where $Z_{D,i}$ are standard Brownian motions[1] with equal pairwise correlation coefficients ρ_D. The aggregate dividend is defined as $D_{M,t} = \sum_{i=1}^{3} D_{i,t}$ and the index dividend is

[1] $Z_{D,i}$ are linear combinations of the fundamental independent Brownian motions Z_i defined in B.1.1. For more details about the transformation, see Appendix C.

defined as $D_{I,t} = \sum_{i=1}^{2} D_{i,t}$.[2] The initial bond value is normalized to unity so that the bond price process is given by

$$B_t = \exp\left(\int_0^t r_s ds\right). \tag{A.2}$$

The stock price processes can be defined as

$$dS_{i,t} = (S_{i,t}\mu_{i,t} - D_{i,t})dt + S_{i,t}\sigma_{i,t}dZ_t, \tag{A.3}$$

where μ_t is the 3-dimensional column vector with $\mu_{i,t}$ as the ith element and σ_t is the 3×3 matrix with $\sigma_{i,t}$ as the ith column. The instantaneous covariance matrix is $\Sigma_t = \sigma_t \sigma_t'$. Both μ_t and σ_t are determined endogenously in equilibrium.

Stock return processes can be defined from stock prices and dividends as

$$dR_{i,t} = \frac{dS_{i,t} + D_{i,t}dt}{S_{i,t}} = \mu_{i,t}dt + \sigma_{i,t}dZ_t. \tag{A.4}$$

The supply of each stock is normalized to one share, while the bond is in zero net supply. There exist a value-weighted index with stocks 1 and 2 as its constituents. The third stock is a non-index stock.

B.1.4 The index and the market

The basket of stocks 1 and 2 is called the index I, which by construction represents a value-weighted index, and the basket of stocks 1, 2 and 3 is called the market M. The index and market baskets therefore also pay dividend streams with dynamics as

[2]See Appendix B.1 for details on the market and the index portfolios.

described in (A.1), with the exception that their variance parameters have the form:

$$\sigma_{D_I} = \left[1 - \frac{2s_1s_2}{(s_1+s_2)^2}(1-\rho_D)\right]\sigma_D^2, \tag{A.5}$$

$$\sigma_{D_M} = \left[1 - 2(s_1s_2 + s_1s_3 + s_2s_3)(1-\rho_D)\right]\sigma_D^2, \tag{A.6}$$

where s_i is the weight of share of dividends of asset i:

$$s_i = \frac{D_i}{D_1 + D_2 + D_3}, \; i \in \{1,2,3\}. \tag{A.7}$$

Let $\omega_{i,t}$ denote the market weight of stock i at time t such that $\sum_{i=1}^{3}\omega_{i,t} = 1$ and let $\omega_{i,t}^I = \omega_{i,t}/(\omega_{1,t}+\omega_{2,t})$ denote the weight of asset $i \in \{1,2\}$ in the index. Then the index return moments are

$$\mu_{I,t} = \omega_{1,t}^I\mu_{1,t} + \omega_{2,t}^I\mu_{2,t}, \tag{A.8}$$

$$\sigma_{I,t}^2 = (\omega_{1,t}^I)^2\sigma_{1,t}^2 + (\omega_{2,t}^I)^2\sigma_{2,t}^2 + 2\omega_{1,t}^I\omega_{2,t}^I corr(dZ_{1,t}, dZ_{2,t})\sigma_{1,t}\sigma_{2,t}. \tag{A.9}$$

B.1.5 Trading strategies

Trading takes place continuously. An admissible trading strategy is a 4-dimensional vector process (α, γ), where γ is an 3-dimensional column vector with γ_i as its ith element and α_t and $\gamma_{i,t}$ denote the amounts invested at time t in the bond and in stock i, satisfying the required regularity conditions.[3]

A trading strategy (α, γ) is said to finance the consumption plan $c \in \mathcal{C}$ if the corresponding wealth process $W = \alpha + \mathbf{1}'\gamma$ satisfies the dynamic budget constraint

$$dW_t = [\alpha_t r_t + \gamma_t'\mu_t - c_t]dt + [\gamma_t'\sigma_t]dZ_t, \tag{A.10}$$

[3]See pp.234-235 of Back (2010) for a formal presentation of the required regularity conditions.

where $\mathbf{1}$ is a 3-dimensional column vector of ones.

B.1.6 Agent's preferences and endowments

There are two representative agents, an active investor $\mathcal{A}$ and an indexer $\mathcal{I}$, both with time-additive log-normal utility functions:

$$U_{j,t}(c) = E_t\left[\int_0^\infty e^{-\delta\tau}\log(c_{j,(t+\tau)})d\tau\right],\ j \in \{\mathcal{A}, \mathcal{I}\} \tag{A.11}$$

for some common rate of time preference $\delta > 0$ and individual consumption c_j.

Agents differ by their endowment and by their investment opportunity set. The indexer is endowed with a fraction β of each index stocks while the active investor owns $(1-\beta)$ share of each index stock and one share of the non-index stock. The indexer faces an exogenous constraint that limits her investment opportunity set to the bond and the index portfolio, according to index weights, which are endogenous. The active investor is unconstrained and faces a complete market.

B.1.7 Equilibrium

Let $\mathcal{E} = ((\Omega, \mathcal{F}, \mathbf{F}, \mathcal{P}), D_1, D_2, D_3, U_1, U_2, \beta)$ denote the primitives for the economy. An equilibrium for the economy $\mathcal{E}$ is an interest rate stock price process (r, S) and a set $\{c_j^*, (\alpha_j^*, \gamma_j^*)\}$, $j \in \{\mathcal{A}, \mathcal{I}\}$ of consumption and admissible trading strategies for the two agents such that:

1. (α_j^*, γ_j^*) finances c_j^* for $j \in \{\mathcal{A}, \mathcal{I}\}$;
2. $c_{\mathcal{A}}^*$ maximizes $U_{\mathcal{A}}$ over the set of consumption plans $c \in \mathcal{C}$ financed by an admissible trading strategy $(\alpha, \gamma) \in \gamma$ with $\alpha_0 + \gamma_0'\mathbf{1} = (1-\beta)[S_{1,0} + S_{2,0}] + S_{3,0}$;

3. $c^*_{\mathcal{I}}$ maximizes $U_{\mathcal{I}}$ over the set on consumption plans $c \in \mathcal{C}$ financed by an admissible trading strategy $(\alpha, \gamma) \in \gamma$ with $\alpha_0 + \gamma_0'\mathbf{1} = \beta[S_{1,0} + S_{2,0}]$, $\gamma_{j,t} = \gamma_{I,t}\frac{S_{i,t}}{S_{1,t}+S_{2,t}}$ for $i = 1, 2$ where γ_I is the amount invested in the index and $\gamma_3 \equiv 0$;

4. all markets clear: $c^*_{\mathcal{A}} + c^*_{\mathcal{I}} = D$, $\alpha^*_{\mathcal{A}} + \alpha^*_{\mathcal{I}} = 0$ and $\gamma^*_{\mathcal{A}} + \gamma^*_{\mathcal{I}} = S$.

B.2 Agents' problem

Agent j's optimization problem at time t is to maximize her time additive utility:

$$U_{j,t} = E_t\left[\int_t^\infty e^{-\delta(s-t)} \log c_{j,s} ds\right] \tag{A.12}$$

subject to her budget constraint. Formally, this gives:

$$\max U_{j,t} \text{ subject to } \; E_t\left[\int_0^\infty \frac{\xi_{j,s}}{\xi_{j,t}} c_{j,s} ds\right] \leq W_{j,t}, \tag{A.13}$$

where $\xi_{j,t}$ is the marginal utility of agent j at time t. The first order condition is:

$$\kappa_j \frac{\xi_{j,s}}{\xi_{j,t}} = e^{-\delta(s-t)} c_{j,s}^{-1}, \tag{A.14}$$

where κ_j is the Lagrange multiplier on the budget constraint and $\xi_{j,t}$ is a process given by:

$$\frac{d\xi_{j,t}}{\xi_{j,t}} = -r_{j,t}dt - \theta'_{j,t} dZ_t. \tag{A.15}$$

where $\theta_{j,t}$ is the price of risk process for agent j. Note that the process can also be written with respect to the dividend basis and the market basis[4] as:

$$\frac{d\xi_{j,t}}{\xi_{j,t}} = -r_{j,t}dt - \overline{\theta}'_{j,t} d\overline{Z}_{D,t} = -r_{j,t}dt - \underline{\theta}'_{j,t} d\underline{Z}_t. \tag{A.16}$$

[4]For a definition of the different bases, see Appendix C.

The rationale for using two different bases, in addition to the initial Brownian motions Z, is that each of the two new bases simplifies the derivation of the solution for a part of the problem and involves independent Brownian motions, which are easier to deal with. It is simpler to solve for optimal portfolios and market clearing under the market basis. However, the market basis transformation depends on stock return covariances, so it is not appropriate to solve for equilibrium price dynamics. The dividend basis is more useful for that purpose.

Since both agents trade in the bond, in equilibrium they should have the same riskless rate (i.e. $r_{\mathcal{I},t} = r_{\mathcal{A},t} = r_t$.) However their different investment opportunity sets means they will face different market price of risk. Following the convex duality methodology approach of Cvitanić and Karatzas (1992), I define a fictitious market which the indexer views as complete. In the current setup with log ulity, the market price of risk in the fictitious market is the same as in the incomplete market (see Example 7.2 on p.304 Karatzas and Shreve (1998) for more details.) The idea is to create a fictitious market for agent $\mathcal{I}$ by replacing the expected return on asset i by $\mu_i(\psi) = \mu_i + \psi_i$ such that in equilibrium she chooses not to hold the unavailable asset, and to hold the index assets according to index weights. In the present setup,

$$\psi = \text{argmin}_{\psi} \left[(\mu_1(\psi) - r, \mu_2(\psi) - r, \mu_3(\psi) - r)\Sigma^{-1}(\mu_1(\psi) - r, \mu_2(\psi) - r, \mu_3(\psi) - r)' \right]^{1/2}. \tag{A.17}$$

Substituting the ψ obtained in (A.17) in the shadow market price of risk of the indexer

I obtain, under the market basis:

$$\underline{\theta}_{\mathcal{I}} = \phi_I \sigma_I^{-1} \begin{bmatrix} \sigma_1 \omega_1^I + \rho_{12}\sigma_2\omega_2^I \\ \sqrt{1-\rho_{12}^2}\sigma_2\omega_2^I \\ 0 \end{bmatrix}, \tag{A.18}$$

where $\phi_I = \frac{\mu_I - r}{\sigma_I}$ is the Sharpe ratio of the index. Since $(\sigma_1\omega_1^I + \rho_{12}\sigma_2\omega_2^I)^2 + (\sqrt{1-\rho_{12}^2}\sigma_2\omega_2^I)^2 = \sigma_I^2$, in scalar form $\theta_{\mathcal{I}} = \phi_I$. The result in (A.18) has the same form if working under the dividend basis following (A.16):

$$\overline{\theta}_{\mathcal{I}} = \phi_I \sigma_I^{-1} \begin{bmatrix} \omega_1^I \overline{\sigma}_{11} + \omega_2^I \overline{\sigma}_{21} \\ \omega_1^I \overline{\sigma}_{12} + \omega_2^I \overline{\sigma}_{22} \\ \omega_1^I \overline{\sigma}_{13} + \omega_2^I \overline{\sigma}_{23} \end{bmatrix}. \tag{A.19}$$

Agent $\mathcal{A}$ is unconstrained and faces complete markets, so her market price of risk under the market and dividend bases are given by:

$$\begin{aligned} \underline{\theta}_{\mathcal{A}} &= \underline{\sigma}^{-1}(\mu_1 - r,\ \mu_2 - r,\ \mu_3 - r)' \\ &= \begin{bmatrix} \phi_1 \\ \frac{\phi_1 - \rho_{12}\phi_2}{\sqrt{1-\phi_{12}^2}} \\ \frac{\phi_3(1-\rho_{12}^2) - \phi_1(\rho_{13}-\rho_{12}\rho_{23}) - \phi_2(\rho_{23}-\rho_{12}\rho_{13})}{\sqrt{1-\rho_{12}^2}\sqrt{1-\rho_{12}^2-\rho_{13}^2-\rho_{23}^2+2\rho_{12}\rho_{13}\rho_{23}}} \end{bmatrix}, \end{aligned} \tag{A.20}$$

$$\begin{aligned} \overline{\theta}_{\mathcal{A}} &= \overline{\sigma}^{-1}(\mu_1 - r,\ \mu_2 - r,\ \mu_3 - r)' \\ &= \frac{1}{c} \begin{bmatrix} x_1(\overline{\sigma}_{23}\overline{\sigma}_{32} - \overline{\sigma}_{22}\overline{\sigma}_{33}) + x_2(\overline{\sigma}_{12}\overline{\sigma}_{33} - \overline{\sigma}_{13}\overline{\sigma}_{32}) + x_3(\overline{\sigma}_{13}\overline{\sigma}_{22} - \overline{\sigma}_{12}\overline{\sigma}_{23}) \\ x_1(\overline{\sigma}_{21}\overline{\sigma}_{33} - \overline{\sigma}_{23}\overline{\sigma}_{31}) + x_2(\overline{\sigma}_{13}\overline{\sigma}_{31} - \overline{\sigma}_{11}\overline{\sigma}_{33}) + (x_3\overline{\sigma}_{11}\overline{\sigma}_{23} - \overline{\sigma}_{13}\overline{\sigma}_{21}) \\ x_1(\overline{\sigma}_{22}\overline{\sigma}_{31} - \overline{\sigma}_{21}\overline{\sigma}_{32}) + x_2(\overline{\sigma}_{11}\overline{\sigma}_{32} - \overline{\sigma}_{12}\overline{\sigma}_{31}) + x_3(\overline{\sigma}_{12}\overline{\sigma}_{21} - \overline{\sigma}_{11}\overline{\sigma}_{22}) \end{bmatrix}, \end{aligned} \tag{A.21}$$

where

$$c = \overline{\sigma}_{13}(\overline{\sigma}_{22}\overline{\sigma}_{31} - \overline{\sigma}_{21}\overline{\sigma}_{32}) + \overline{\sigma}_{12}(\overline{\sigma}_{21}\overline{\sigma}_{33} - \overline{\sigma}_{23}\overline{\sigma}_{31}) + \overline{\sigma}_{11}(\overline{\sigma}_{23}\overline{\sigma}_{32} - \overline{\sigma}_{22}\overline{\sigma}_{33}),$$

and $x_i = \mu_i - r$ is the excess return on asset i.

B.3 Optimal portfolios

Agent $\mathcal{A}$ is unconstrained, so her optimal portfolio proportions are given by

$$\pi_{\mathcal{A},t} = \Sigma_t^{-1}(\mu_t - r\mathbf{1}). \tag{A.22}$$

Under the market basis the covariance matrix is $\Sigma_t = \underline{\sigma}_t \underline{\sigma}_t'$, so

$$\pi_{\mathcal{A}} = \begin{bmatrix} \frac{\phi_1(1-\rho_{23}^2)-\phi_2(\rho_{12}-\rho_{13}\rho_{23})-\phi_3(\rho_{13}-\rho_{12}\rho_{23})}{\sigma_1(1-\rho_{12}^2-\rho_{13}^2-\rho_{23}^2+2\rho_{12}\rho_{13}\rho_{23})} \\ \frac{\phi_2(1-\rho_{13}^2)-\phi_1(\rho_{12}-\rho_{13}\rho_{23})-\phi_3(\rho_{23}-\rho_{12}\rho_{13})}{\sigma_2(1-\rho_{12}^2-\rho_{13}^2-\rho_{23}^2+2\rho_{12}\rho_{13}\rho_{23})} \\ \frac{\phi_3(1-\rho_{12}^2)-\phi_1(\rho_{13}-\rho_{12}\rho_{23})-\phi_2(\rho_{23}-\rho_{12}\rho_{13})}{\sigma_3(1-\rho_{12}^2-\rho_{13}^2-\rho_{23}^2+2\rho_{12}\rho_{13}\rho_{23})} \end{bmatrix}. \tag{A.23}$$

As for agent $\mathcal{I}$, I know from Cvitanić and Karatzas (1992) that $\pi_{\mathcal{I},t}$ coincides with the optimal portfolio in the incomplete market:

$$\pi_{\mathcal{I}} = \begin{bmatrix} \pi_{\mathcal{I}}^I \omega_1^I \\ \pi_{\mathcal{I}}^I \omega_2^I \\ 0 \end{bmatrix}, \tag{A.24}$$

where $\pi^I_{\mathcal{I},t} = (\mu_{I,t} - r)/\sigma^2_{I,t}$, so

$$\pi_{\mathcal{I}} = \begin{bmatrix} \omega_1^I \frac{\phi_I}{\sigma_I} \\ \omega_2^I \frac{\phi_I}{\sigma_I} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\omega_1^I\left(x_1\omega_1^I + x_2\omega_2^I\right)}{\sigma_1^2(\omega_1^I)^2 + 2\rho_{12}\sigma_1\sigma_2\omega_1^I\omega_2^I + \sigma_2^2(\omega_2^I)^2} \\ \frac{\omega_2^I\left(x_1\omega_1^I + x_2\omega_2^I\right)}{\sigma_1^2(\omega_1^I)^2 + 2\rho_{12}\sigma_1\sigma_2\omega_1^I\omega_2^I + \sigma_2^2(\omega_2^I)^2} \\ 0 \end{bmatrix}. \tag{A.25}$$

The market clearing condition imposes that

$$\omega_t = \pi_{\mathcal{A},t}\nu_{\mathcal{A},t} + \pi_{\mathcal{I},t}\nu_{\mathcal{I},t}, \tag{A.26}$$

where ω_t is a 3-dimensional vector with the i-th element equal to the value-weight of stock i in the economy ($\omega_i = S_i / \sum_{k=1}^3 S_k$). Substituting the optimal portfolio weights in the market clearing condition yields the following proposition:

Proposition A.1 *In equilibrium, expected excess stock returns are as follows:*

$$\begin{aligned} \mu_1 - r_f = \frac{1}{\sigma_I^2\omega_I} &[(\mu_I - r_f)(\sigma_1\omega_1 + \rho_{1,2}\sigma_2\omega_2) \\ &+ \frac{\omega_2\omega_3}{\nu_A\omega_I}\Big(\omega_1[cov(dR_1, dR_2)cov(dR_1, dR_3) - \sigma_1^2 cov(dR_2, dR_3)] \\ &-\omega_2[cov(dR_1, dR_2)cov(dR_2, dR_3) - \sigma_2^2 cov(dR_1, dR_3)]\Big)\Big], \end{aligned} \tag{A.27}$$

$$\mu_3 - r_f = \omega_3\sigma_3^2 + (1-\omega_3)cov(dR_I, dR_3) + \omega_3\sigma_3^2\left[\frac{\nu_I}{\nu_A}(1-\rho_{I,3}^2)\right], \tag{A.28}$$

$$\mu_I - r_f = \omega_I\sigma_I^2 + (1-\omega_I)cov(dR_I, dR_3), \tag{A.29}$$

where μ_I and σ_I denote the drift and variance of the index and $\omega_I = \omega_1 + \omega_2$. Result *for stock 2 is omitted as it is symmetric to stock 1.*

Proof The market clearing condition imposes that:

$$
\begin{aligned}
\omega_t &= \pi_{\mathcal{A},t}\nu_{\mathcal{A},t} + \pi_{\mathcal{I},t}\nu_{\mathcal{I},t} \\
&= \begin{bmatrix}
\frac{\nu_{\mathcal{A}}\left(x_3(\rho_{13}-\rho_{12}\rho_{23})\sigma_1\sigma_2+\left(x_2(\rho_{12}-\rho_{13}\rho_{23})\sigma_1+x_1\left(-1+\rho_{23}^2\right)\sigma_2\right)\sigma_3\right)}{\left(-1+\rho_{12}^2+\rho_{13}^2-2\rho_{12}\rho_{13}\rho_{23}+\rho_{23}^2\right)\sigma_1^2\sigma_2\sigma_3} - \frac{(-1+\nu_{\mathcal{A}})\omega_1(x_1\omega_1+x_2\omega_2)}{\sigma_1^2\omega_1^2+2\rho_{12}\sigma_1\sigma_2\omega_1\omega_2+\sigma_2^2\omega_2^2} \\
\frac{\nu_{\mathcal{A}}\left(x_3(-\rho_{12}\rho_{13}+\rho_{23})\sigma_1\sigma_2+\left(x_2\left(-1+\rho_{13}^2\right)\sigma_1+x_1(\rho_{12}-\rho_{13}\rho_{23})\sigma_2\right)\sigma_3\right)}{\left(-1+\rho_{12}^2+\rho_{13}^2-2\rho_{12}\rho_{13}\rho_{23}+\rho_{23}^2\right)\sigma_1\sigma_2^2\sigma_3} - \frac{(-1+\nu_{\mathcal{A}})\omega_2(x_1\omega_1+x_2\omega_2)}{\sigma_1^2\omega_1^2+2\rho_{12}\sigma_1\sigma_2\omega_1\omega_2+\sigma_2^2\omega_2^2} \\
\frac{\nu_{\mathcal{A}}\left(x_3\left(-1+\rho_{12}^2\right)\sigma_1\sigma_2+(x_2(-\rho_{12}\rho_{13}+\rho_{23})\sigma_1+x_1(\rho_{13}-\rho_{12}\rho_{23})\sigma_2)\sigma_3\right)}{\left(-1+\rho_{12}^2+\rho_{13}^2-2\rho_{12}\rho_{13}\rho_{23}+\rho_{23}^2\right)\sigma_1\sigma_2\sigma_3^2}
\end{bmatrix},
\end{aligned}
\tag{A.30}
$$

where $x_i = \mu_i - r$ are excess returns. Solving for x_1, x_2 and x_3, I get:

$$
\begin{aligned}
x_1^* = &\left(\sigma_1 \left(\sigma_2\sigma_3\omega_2\omega_3 \left(\rho_{12}\rho_{13}\sigma_1\omega_1 - \rho_{23}\sigma_1\omega_1 + \rho_{13}\sigma_2\omega_2 - \rho_{12}\rho_{23}\sigma_2\omega_2\right)\right.\right. \\
&\left.\left. +\nu_{\mathcal{A}} \left(\sigma_1\omega_1 + \rho_{12}\sigma_2\omega_2\right) \left(\sigma_1^2\omega_1^2 + 2\rho_{12}\sigma_1\sigma_2\omega_1\omega_2 + \rho_{13}\sigma_1\sigma_3\omega_1\omega_3 + \sigma_2^2\omega_2^2 + \rho_{23}\sigma_2\sigma_3\omega_2\omega_3\right)\right)\right) \\
&/ \left(\nu_{\mathcal{A}} \left(\sigma_1^2\omega_1^2 + 2\rho_{12}\sigma_1\sigma_2\omega_1\omega_2 + \sigma_2^2\omega_2^2\right)\right) \\
= &\left(\omega_1\sigma_1 + \omega_2\rho_{12}\sigma_2\right) \left(1 - \frac{\omega_3\sigma_3}{\sigma_I^2}(\omega_1\rho_{13}\sigma_1 + \omega_2\rho_{23}\sigma_2\right) \\
&+ \frac{\omega_2\omega_3\sigma_2\sigma_3}{\nu_{\mathcal{A}}\sigma_I^2} \left[\omega_1\sigma_1(\rho_{12}\rho_{13} - \rho_{23}) - \omega_2\sigma_2(\rho_{12}\rho_{23} - \rho_{13})\right].
\end{aligned}
\tag{A.31}
$$

I can also write x_1^* in terms of x_I^*:

$$
\begin{aligned}
x_1^* = &\frac{1}{\nu_A \sigma_I^2 \omega_I^2} \{\sigma_1 (\sigma_2\sigma_3\omega_2\omega_3 (\rho_{12}\rho_{13}\sigma_1\omega_1 - \rho_{23}\sigma_1\omega_1 + \rho_{13}\sigma_2\omega_2 - \rho_{12}\rho_{23}\sigma_2\omega_2) \\
&+\nu_{\mathcal{A}} (\sigma_1\omega_1 + \rho_{12}\sigma_2\omega_2) (x_I^*\omega_I))\} \\
= &\frac{1}{\sigma_I^2\omega_I} [x_I^*(\sigma_1\omega_1 + \rho_{1,2}\sigma_2\omega_2) \\
&+ \frac{\omega_2\omega_3}{\nu_A} \left(\frac{\omega_1}{\omega_I}[cov(dR_1, dR_2)cov(dR_1, dR_3) - \sigma_1^2 cov(dR_2, dR_3)]\right. \\
&\left.-\frac{\omega_2}{\omega_I}[cov(dR_1, dR_2)cov(dR_2, dR_3) - \sigma_2^2 cov(dR_1, dR_3)]\right)\Bigg]. \qquad \text{(A.32)}
\end{aligned}
$$

For x_3^*, I get

$$
\begin{aligned}
x_3^* = &\Big(\sigma_3 \Big(\nu_{\mathcal{A}}\rho_{13}\sigma_1^3\omega_1^3 + \sigma_1^2\omega_1^2 \Big(\nu_{\mathcal{A}} (2\rho_{12}\rho_{13} + \rho_{23}) \sigma_2\omega_2 + \Big(1 + (-1 + \nu_{\mathcal{A}}) \rho_{13}^2\Big) \sigma_3\omega_3\Big) \\
&+\sigma_2^2\omega_2^2 \Big(\nu_{\mathcal{A}}\rho_{23}\sigma_2\omega_2 + \Big(1 + (-1 + \nu_{\mathcal{A}}) \rho_{23}^2\Big) \sigma_3\omega_3\Big) \\
&+\sigma_1\sigma_2\omega_1\omega_2 (2\rho_{12} (\nu_{\mathcal{A}}\rho_{23}\sigma_2\omega_2 + \sigma_3\omega_3) + \rho_{13} (\nu_{\mathcal{A}}\sigma_2\omega_2 + 2 (-1 + \nu_{\mathcal{A}}) \rho_{23}\sigma_3\omega_3)))) \\
&/ \Big(\nu_{\mathcal{A}} \Big(\sigma_1^2\omega_1^2 + 2\rho_{12}\sigma_1\sigma_2\omega_1\omega_2 + \sigma_2^2\omega_2^2\Big)\Big) \\
= &\,\omega_I cov(dR_I, dR_3) + \omega_3\sigma_3^2 \left[1 + \frac{\nu_I}{\nu_A}(1 - \rho_{I,3}^2)\right], \qquad \text{(A.33)}
\end{aligned}
$$

where

$$
\begin{aligned}
x_I^* &= \frac{\sigma_1^2\omega_1^2 + 2\rho_{12}\sigma_1\sigma_2\omega_1\omega_2 + \sigma_2^2\omega_2^2 + \rho_{13}\sigma_1\sigma_3\omega_1\omega_3 + \rho_{23}\sigma_2\sigma_3\omega_2\omega_3}{\omega_1 + \omega_2} \\
&= \sigma_I^2\omega_I + \omega_3 cov(dR_I, dR_3), \qquad \text{(A.34)}
\end{aligned}
$$

with $\omega_I = \omega_1 + \omega_2$. Results for x_2 are omitted as they are symmetric to x_1.

Proposition A.1 tells us that holding variances and covariances constant, the non-index stock excess returns are increasing in the relative wealth of passive investors. This is due to the additional risk active investors are taking as they become more under-diversified (compared with the case where they hold the market). It is highlighted by the term of correlation between the index and stock 3. This is consistent with the standard result from one period models of mild segmentation (see, e.g., Errunza and Losq (1985).) I cannot however conclude from Proposition A.1 on the actual equilibrium effect of an increase in $\nu_{\mathcal{I}}$ as the variance and covariance terms are determined endogenously in equilibrium and thus also depend on $\nu_{\mathcal{I}}$.

B.4 Proof of Proposition 1

Following Cuoco and He (1994), I can still use a social planner to derive equilibrium prices, but the weight λ_t will be stochastic:

$$U_t = E_t \int_t^{\infty} e^{-\delta(s-t)} \left(\log c_{\mathcal{A},s} + \lambda_s \log c_{\mathcal{I},s}\right) ds. \tag{A.35}$$

The consumption sharing rule is given by:

$$1 = \frac{c_{\mathcal{A},t}^{-1}}{\lambda_t c_{\mathcal{I},t}^{-1}}. \tag{A.36}$$

I define Agent j's equilibrium share of world consumption as $\nu_{j,t} = \frac{c_{j,t}}{D_{M,t}}$. In equilibrium the two agents must consume the aggregate dividend: $c_{\mathcal{A},t} + c_{\mathcal{I},t} = D_{M,t}$. Thus,

$$\nu_{\mathcal{A},t} = \frac{1}{1+\lambda_t}, \qquad \nu_{\mathcal{I},t} = \frac{\lambda_t}{1+\lambda_t}. \tag{A.37}$$

As in Basak and Cuoco (1998), the equilibrium state-price density ξ_t is given by the state-price density of the unconstrained agent $\mathcal{A}$:

$$\xi_t = \xi_{\mathcal{A},t} = \kappa_{\mathcal{A}} e^{-\delta t}(\nu_{\mathcal{A},t} D_{M,t})^{-1}. \tag{A.38}$$

To solve for equilibrium prices, I need to derive an expression λ_t and the related process $\nu_{\mathcal{A},t}$. Sustituting $c_{\mathcal{A}}$ and $c_{\mathcal{I}}$ from (A.14) in (A.36), I get:

$$\lambda_t = \frac{\kappa_{\mathcal{A}} \xi_{\mathcal{A},t}/\xi_{\mathcal{A},0}}{\kappa_{\mathcal{I}} \xi_{\mathcal{I},t}/\xi_{\mathcal{I},0}}. \tag{A.39}$$

Solving (A.16), agent j's state-price density under the dividend basis, gives:

$$\xi_{j,t} = \xi_{j,0} e^{-\int_0^t (r_s + \frac{1}{2}\theta_{j,s}^2) ds - \int_0^t \overline{\theta}'_{j,s} d\overline{Z}_{D,s}} \tag{A.40}$$

where $\theta_{j,s} = \overline{\theta}'_{j,s}\mathbf{1}$ and $\mathbf{1}$ is a vector of ones. Substituting (A.40) in (A.39) gives:

$$\lambda_t = \frac{\kappa_{\mathcal{A}}}{\kappa_{\mathcal{I}}} e^{-\int_0^t \frac{1}{2}(\theta_{\mathcal{A},s}^2 - \theta_{\mathcal{I},s}^2) ds - \int_0^t (\overline{\theta}_{\mathcal{A},s} - \overline{\theta}_{\mathcal{I},s})' d\overline{Z}_{D,s}}. \tag{A.41}$$

Applying Itô's Lemma gives:

$$\frac{d\lambda_t}{\lambda_t} = \mu_{\lambda,t} dt + \overline{\sigma}'_{\lambda,t} d\overline{Z}_{D,t}, \tag{A.42}$$

where

$$\mu_{\lambda,t} = \overline{\theta}'_{\mathcal{I},t}(\overline{\theta}_{\mathcal{I},t} - \overline{\theta}_{\mathcal{A},t}), \tag{A.43}$$

$$\overline{\sigma}_{\lambda,t} = (\overline{\theta}_{\mathcal{I},t} - \overline{\theta}_{\mathcal{A},t}). \tag{A.44}$$

Rewriting as a scalar process, I get:

$$\frac{d\lambda_t}{\lambda_t} = \mu_{\lambda,t}dt + \sigma_{\lambda,t}dZ_{\lambda,t}, \tag{A.45}$$

where

$$\sigma_{\lambda,t} = \sqrt{(\overline{\theta}_{\mathcal{I},t} - \overline{\theta}_{\mathcal{A},t})'(\overline{\theta}_{\mathcal{I},t} - \overline{\theta}_{\mathcal{A},t})}, \tag{A.46}$$

$$dZ_{\lambda,t} = \sigma_{\lambda,t}^{-1}\overline{\sigma}'_{\lambda,t}d\overline{Z}_{D,t}. \tag{A.47}$$

Remember that:

$$\overline{\theta}_{\mathcal{I}} = \frac{x_I}{\sigma_I^2}\overline{\sigma}'\begin{bmatrix}\omega_1^I\\ \omega_2^I\\ 0\end{bmatrix}, \qquad \overline{\theta}_{\mathcal{A}} = \overline{\sigma}^{-1}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}.$$

Therefore,

$$\overline{\theta}_{\mathcal{I}} - \overline{\theta}_{\mathcal{A}} = \frac{x_I}{\sigma_I^2}\overline{\sigma}'\begin{bmatrix}\omega_1^I\\ \omega_2^I\\ 0\end{bmatrix} - \overline{\sigma}^{-1}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}, \tag{A.48}$$

$$\begin{aligned}\overline{\sigma}(\overline{\theta}_{\mathcal{I}} - \overline{\theta}_{\mathcal{A}}) &= \frac{x_I}{\sigma_I^2}\Sigma\begin{bmatrix}\omega_1^I\\ \omega_2^I\\ 0\end{bmatrix} - \begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}\\ &= \begin{bmatrix}\frac{\omega_2^I}{\sigma_I^2}[x_2\omega_2^I(\omega_1^I\sigma_1^2 + \omega_2^I\rho_{12}\sigma_1\sigma_2) - x_1\omega_2^I(\omega_2^I\sigma_2^2 + \omega_1^I\rho_{12}\sigma_1\sigma_2)]\\ \frac{\omega_1^I}{\sigma_I^2}[x_1\omega_1^I(\omega_2^I\sigma_2^2 + \omega_1^I\rho_{12}\sigma_1\sigma_2) - x_2\omega_1^I(\omega_1^I\sigma_1^2 + \omega_2^I\rho_{12}\sigma_1\sigma_2)]\\ x_I\beta_{I,3} - x_3\end{bmatrix},\end{aligned} \tag{A.49}$$

where $\beta_{I,3} = \rho_{I,3}\sigma_3/\sigma_I = (\omega_1^I\rho_{13}\sigma_1\sigma_3 + \omega_2^I\rho_{23}\sigma_2\sigma_3)/\sigma_I^2$. One can easily see that $\overline{\theta}_{\mathcal{I}}'\overline{\theta}_{\mathcal{I}} = \frac{x_I^2}{\sigma_I^2}$ and that $\overline{\theta}_{\mathcal{I}}'\overline{\theta}_{\mathcal{A}} = \frac{x_I^2}{\sigma_I^2}$. Note that those results are basis invariant. I obtain:

$$\mu_\lambda = \overline{\theta}_{\mathcal{I}}'(\overline{\theta}_{\mathcal{I}} - \overline{\theta}_{\mathcal{A}}) = 0. \tag{A.50}$$

Similarly,

$$\begin{aligned}\sigma_\lambda^2 &= (\overline{\theta}_{\mathcal{I}} - \overline{\theta}_{\mathcal{A}})'(\overline{\theta}_{\mathcal{I}} - \overline{\theta}_{\mathcal{A}}) \\ &= -\frac{x_I^2}{\sigma_I^2} + \overline{\theta}_{\mathcal{A}}'\overline{\theta}_{\mathcal{A}}, \qquad \text{(A.51)} \\ \Rightarrow \sigma_\lambda &= \sqrt{[x_1\ x_2\ x_3]\Sigma^{-1}[x_1\ x_2\ x_3]' - \frac{x_I^2}{\sigma_I^2}}. \qquad \text{(A.52)}\end{aligned}$$

Using the definition of $\nu_{\mathcal{A}}$ in (A.37) and applying Itô Lemma gives:

$$d\nu_{\mathcal{A}} = \mu_{\nu_{\mathcal{A}}}dt + \overline{\sigma}_{\nu_{\mathcal{A}}}'d\overline{Z}_D, \tag{A.53}$$

where

$$\mu_{\nu_{\mathcal{A}}} = \nu_{\mathcal{A}}\nu_{\mathcal{I}}^2\sigma_\lambda^2, \tag{A.54}$$

$$\overline{\sigma}_{\nu_{\mathcal{A}}} = \nu_{\mathcal{A}}\nu_{\mathcal{I}}\overline{\sigma}_\lambda. \tag{A.55}$$

In scalar notation this becomes:

$$d\nu_{\mathcal{A}} = \mu_{\nu_{\mathcal{A}}}dt + \sigma_{\nu_{\mathcal{A}}}dZ_\lambda, \tag{A.56}$$

$$\sigma_{\nu_{\mathcal{A}}} = -\nu_{\mathcal{A}}\nu_{\mathcal{I}}\sigma_\lambda. \tag{A.57}$$

Applying Itô's Lemma to (A.38), I obtain:

$$\frac{d\xi}{\xi} = -\left[\delta + \mu_{D_M} - \sigma_{D_M}^2 + \frac{\rho_{\nu_{\mathcal{A}} D_M} \sigma_{\nu_{\mathcal{A}}} \sigma_{D_M}}{\nu_{\mathcal{A}}}\right] dt - \left[\overline{\sigma}'_{D_M} + \frac{\overline{\sigma}'_{\nu_{\mathcal{A}}}}{\nu_{\mathcal{A}}}\right] d\overline{Z}_D. \tag{A.58}$$

Equaling the terms to those in (A.16), I get:

$$r_f = \delta + \mu_{D_M} - \sigma_{D_M}^2 + \frac{\rho_{\nu_{\mathcal{A}} D_M} \sigma_{\nu_{\mathcal{A}}} \sigma_{D_M}}{\nu_{\mathcal{A}}}, \tag{A.59}$$

$$\overline{\theta} = \overline{\sigma}_{D_M} + \frac{\overline{\sigma}_{\nu_{\mathcal{A}}}}{\nu_{\mathcal{A}}}. \tag{A.60}$$

B.5 Proof of Corollary 1

From (A.38) I can assert that $\overline{\theta} = \overline{\theta}_{\mathcal{A}}$. Thus, from (A.44), (A.55) and (A.60),

$$\begin{aligned} \overline{\theta}_{\mathcal{A}} &= \overline{\sigma}_{D_M} + \frac{\overline{\sigma}_{\nu_{\mathcal{A}}}}{\nu_{\mathcal{A}}} \\ &= \overline{\sigma}_{D_M} - \nu_{\mathcal{I}} \overline{\sigma}_{\lambda} \\ &= \overline{\sigma}_{D_M} - \nu_{\mathcal{I}} (\overline{\theta}_{\mathcal{I}} - \overline{\theta}_{\mathcal{A}}), \end{aligned} \tag{A.61}$$

$$\Rightarrow \overline{\theta} = \frac{\overline{\sigma}_{D_M}}{\nu_{\mathcal{A}}} - \frac{\nu_{\mathcal{I}}}{\nu_{\mathcal{A}}} \overline{\theta}_{\mathcal{I}}, \tag{A.62}$$

$$\overline{\theta} = \overline{\sigma}_{D_M} + \frac{\nu_{\mathcal{I}}}{\nu_{\mathcal{A}}} \left(\overline{\sigma}_{D_M} - \overline{\theta}_{\mathcal{I}}\right). \tag{A.63}$$

Note here that $\overline{\sigma}_{D_M}$ is exogenous to the model (when defined relative to the dividend basis), $\nu_{\mathcal{I}}$ and $\nu_{\mathcal{A}} = 1 - \nu_{\mathcal{I}}$ are state variables and the other quantities are determined endogenously in equilibrium. Denoting $\overline{\theta}^{\star} = \overline{\sigma}_{D_M}$ the price of risk when there are no indexers ($\nu_{\mathcal{A}} = 1$, $\nu_{\mathcal{I}} = 0$),

$$\begin{aligned} \overline{\theta} &= \overline{\theta}^{\star} + \frac{\nu_{\mathcal{I}}}{\nu_{\mathcal{A}}} \left(\overline{\theta}^{\star} - \overline{\theta}_{\mathcal{I}} \right) \\ \Rightarrow \left(\overline{\theta}^{\star} - \overline{\theta}_{\mathcal{A}} \right) &= -\frac{\nu_{\mathcal{I}}}{\nu_{\mathcal{A}}} \left(\overline{\theta}^{\star} - \overline{\theta}_{\mathcal{I}} \right). \end{aligned} \tag{A.64}$$

B.6 Corollary A.1 to Proposition 1

Corollary A.1 *In equilibrium, the share of aggregate wealth owned by the active investor follows the process:*

$$d\nu_{\mathcal{A}} = \mu_{\nu_{\mathcal{A}}} dt + \sigma_{\nu_{\mathcal{A}}} dZ_{\lambda}, \tag{A.65}$$

where

$$\mu_{\nu_{\mathcal{A}}} = \nu_{\mathcal{A}} \nu_{\mathcal{I}}^2 \sigma_{\lambda}^2, \tag{A.66}$$

$$\sigma_{\nu_{\mathcal{A}}} = \nu_{\mathcal{A}} \nu_{\mathcal{I}} \sigma_{\lambda}, \tag{A.67}$$

and σ_{λ} is the volatility of the stochastic weight in the representative agent's problem.

Proof Follows from the proof of Proposition 1.

Corollary A.1 illustrates that the equilibrium is not stationary. Since $\mu_{\nu_{\mathcal{A}}}$ is positive, over time the active investor will dominate, and $\nu_{\mathcal{A}} = 1$ ($\nu_{\mathcal{A}} = 0$) is an absorbing state (in that case both $\mu_{\nu_{\mathcal{A}}}$ and $\sigma_{\nu_{\mathcal{A}}}$ are equal to 0). This is a standard feature of models with constrained investors; the unconstrained one will dominate over time since both agents have the same preferences but differ in their investment opportunity sets. Thus, in its current form, the model cannot explain the rise of indexing of the past decades. A richer model could generate the observed level of indexing as an endogenous outcome in a general equilibrium setup. For example, this could be done by adding frictions such as the incremental cost of active investing and allowing one rational agent to invest both passively and actively at the same time, as in Petajisto (2009). However, the additional complexity is not necessary for the current discussion. The current form of the model provides valuable insight on stock prices dynamics for given levels of indexing, which is the purpose of this paper.

B.7 Proof of Proposition 2

In this section, I derive the dynamics of each stock's price process.

Since both agents have time-additive log utility, it follows from the first order condition of the HJB equation that the aggregate stock market value $S_{M,t} = S_{1,t} + S_{2,t} + S_{3,t} = D_{M,t}/\delta$, and thus aggregate stock market value is independent of the relative wealth of agents.[5]

The price $S_{i,t}$ of stock i at time t is the expected value of future dividends discounted using the stochastic discount factor of the representative agent ξ defined in (A.58):

$$S_{i,t} = E_t\left[\int_t^\infty \frac{\xi_\tau}{\xi_t} D_{i,\tau} d\tau\right]. \tag{A.68}$$

Using the results from equations (A.14) and (A.38), I have

$$S_{i,t} = E_t\left[\int_t^\infty e^{-\delta(\tau-t)}\left(\frac{c_{\mathcal{A},\tau}}{c_{\mathcal{A},t}}\right)^{-1} D_{i,\tau} d\tau\right]. \tag{A.69}$$

[5]For infinite horizon log utility, $J(t,w,x) = \frac{\log w}{\delta} + f(t,x)$. The FOC of the HJB equation is $u'(c_i) = J_{w_i}$, which yields $c_i = w_i\delta$. In the current setup, the representative agent must consume the aggregate dividend and own the aggregate stock market, thus $c_{\mathcal{A}} + c_{\mathcal{I}} = D_M$ and $w_{\mathcal{A}} + w_{\mathcal{I}} = S_M$. We thus have that $S_M = D_M/\delta$.

From (A.37), I have:

$$c_{\mathcal{A},t} = \frac{D_{M,t}}{1+\lambda_t}, \tag{A.70}$$

thus

$$\frac{c_{\mathcal{A},t}}{c_{\mathcal{A},\tau}} = \frac{D_{M,t}}{D_{M,\tau}} \frac{1+\lambda_\tau}{1+\lambda_t}. \tag{A.71}$$

Substituting this last result in (A.69), I obtain:

$$\begin{aligned} S_{i,t} &= D_{M,t} E_t \left[\int_t^\infty e^{-\delta(\tau-t)} \frac{1+\lambda_\tau}{1+\lambda_t} s_{i,\tau} d\tau \right] \\ &= D_{M,t} f_{i,t}, \end{aligned} \tag{A.72}$$

where

$$\begin{aligned} f_{i,t} &= E_t \left[\int_t^\infty e^{-\delta(\tau-t)} \frac{1+\lambda_\tau}{1+\lambda_t} s_{i,\tau} d\tau \right] \\ &= \underbrace{\frac{1}{1+\lambda_t}}_{\nu_{\mathcal{A},t}} \underbrace{E_t \left[\int_t^\infty e^{-\delta(\tau-t)} s_{i,\tau} d\tau \right]}_{f_{i,t}^{\mathcal{A}}} + \underbrace{\frac{\lambda_t}{1+\lambda_t}}_{\nu_{\mathcal{I},t}} \underbrace{E_t \left[\int_t^\infty e^{-\delta(\tau-t)} \frac{\lambda_\tau}{\lambda_t} s_{i,\tau} d\tau \right]}_{f_{i,t}^{\mathcal{I}}} \qquad (A.73) \\ &= \nu_{\mathcal{A},t} f_{i,t}^{\mathcal{A}} + \nu_{\mathcal{I},t} f_{i,t}^{\mathcal{I}}. \qquad (A.74) \end{aligned}$$

Note that in a world without constraints, λ_t is constant and we thus have $f_{i,t} = f_{i,t}^{\mathcal{A}}$. Alternatively, I can get this result by setting $\nu_{\mathcal{A},t} = 1$ and $\nu_{\mathcal{I},t} = 0$.

B.8 Solving for $f_{i,t}^{\mathcal{A}}$

$f_{i,t}^{\mathcal{A}}$ depends on the relative share of the aggregate dividend of each stock, $s_{i,t}$ as defined in (A.7). Therefore,

$$s_{i,t} = \frac{D_{i,t}}{D_{M,t}}. \tag{A.75}$$

To fully characterize the relative weights of each dividend stream two of those s_i are sufficient, so I need two state variables. Using Itô's Lemma, I obtain:

$$\frac{ds_i^M}{s_i^M} = \left[\overline{\sigma}'_{D_M}(\overline{\sigma}_{D_M} - \overline{\sigma}_{D_i})\right] dt$$
$$+ (\overline{\sigma}_{D_i} - \overline{\sigma}_{D_M})' d\overline{Z}_D, \tag{A.76}$$

which after simplification yields

$$ds_i = \mu_{s_i} dt + \overline{\sigma}'_{s_i} d\overline{Z}_D, \tag{A.77}$$

where

$$\mu_{s_i} = s_i s_{-i} \left[-s_i \sigma_D^2 + s_{-i} \sigma_{D_{-i}}^2 + (s_i - s_{-i}) \rho_{D_i D_{-i}} \sigma_D \sigma_{D_{-i}}\right], \tag{A.78}$$

$$\overline{\sigma}_{s_i} = s_i s_{-i} (\overline{\sigma}_{D_i} - \overline{\sigma}_{D_{-i}}), \tag{A.79}$$

and D_{-i} represents the dividend stream of the other two stocks combined.

Defining $y_{i,t} = \log \frac{s_{i,t}}{s_{-i,t}}$, it follows from Itô's Lemma that:

$$dy_i = \mu_{y_i} dt + \overline{\sigma}'_{y_i} d\overline{Z}_D \tag{A.80}$$

where

$$\mu_{y_i} = \left[\mu_{D_i} - \frac{1}{2}\sigma_{D_i}^2\right] - \left[\mu_{D_{-i}} - \frac{1}{2}\sigma_{D_{-i}}^2\right], \tag{A.81}$$

$$\overline{\sigma}_{y_i} = \overline{\sigma}_{D_i} - \overline{\sigma}_{D_{-i}}. \tag{A.82}$$

In scalar form,

$$dy_i = \mu_{y_i} dt + \sigma_{y_i} dZ_{y_i}, \tag{A.83}$$

where

$$\sigma_{y_i} = \sqrt{(\overline{\sigma}_{D_i} - \overline{\sigma}_{D_{-i}})'\overline{\sigma}_{D_i} - \overline{\sigma}_{D_{-i}}}$$
$$= \sqrt{\sigma^2_{D_i} + \sigma^2_{D_{-i}} - 2\rho_{D_i D_{-i}}\sigma_{D_i}\sigma_{D_{-i}}}, \tag{A.84}$$

$$Z_{y_i} = \sigma_{y_i}^{-1}\overline{\sigma}'_{y_i} d\overline{Z}_D. \tag{A.85}$$

From Cochrane, Longstaff, and Santa-Clara (2008), I know there is a closed-form expression for $f^{\mathcal{A}}_{i,t}$ if y_i is the only relevant state variable ($\nu_{\mathcal{A}}$ is irrelevant for $f^{\mathcal{A}}_{i,t}$). In the present case the moments of the dividend process of portfolio $-i$ also depend on the relative dividend of the two stocks in that portfolio, i.e. y_1 depends on D_2/D_3. So $f^{\mathcal{A}}_{i,t}$ depends on two state variables representing the relative dividend processes. Let's use y_1 and y_2 as the state variables. Note also that since $\sum_{i=1}^3 f^{\mathcal{A}}_{i,t} = \frac{1}{\delta}$, we only need to solve for two i to get the third one. We'll solve for $i = 1, 2$ so the functions will be symmetric. Here I show the derivation of $f^{\mathcal{A}}_{1,t}$. Note from (A.77) that $s_i = 0$ and $s_i = 1$ are absorbing states, so we obtain the following boundary conditions:

$$\lim_{y_1 \to -\infty} f^{\mathcal{A}}_{1,t} = 0, \tag{A.86}$$

$$\lim_{y_1 \to \infty} f^{\mathcal{A}}_{1,t} = \frac{1}{\delta}, \tag{A.87}$$

$$\lim_{y_2 \to \infty} f^{\mathcal{A}}_{1,t} = 0. \tag{A.88}$$

The boundary condition $\lim_{y_2 \to -\infty} f^{\mathcal{A}}_{1,t}$ is less obvious because in that case asset 2 becomes irrelevant, so $f^{\mathcal{A}}_{1,t}$ converges to the Cochrane, Longstaff, and Santa-Clara (2008) case. From the Feynman-Kac theorem, we can transform the problem to a PDE representation:

$$\frac{1}{2}\sigma_{y_1}^2\frac{\partial^2 f_1^{\mathcal{A}}}{\partial y_1^2}+\frac{1}{2}\sigma_{y_2}^2\frac{\partial^2 f_1^{\mathcal{A}}}{\partial y_2^2}+\overline{\sigma}'_{y_1}\overline{\sigma}_{y_2}\frac{\partial^2 f_1^{\mathcal{A}}}{\partial y_1\partial y_2}+\mu_{y_1}\frac{\partial f_1^{\mathcal{A}}}{\partial y_1}+\mu_{y_2}\frac{\partial f_1^{\mathcal{A}}}{\partial y_2}-\rho f_1^{\mathcal{A}}+\frac{1}{1+e^{-y_1}}=0, \tag{A.89}$$

where

$$\begin{aligned}
\mu_{y_1} &= -\left[\frac{s_2-s_1s_2-s_2^2}{(1-s_1)^2}\right](1-\rho_D)\sigma_D^2,\\
\mu_{y_2} &= -\left[\frac{s_1-s_1s_2-s_1^2}{(1-s_2)^2}\right](1-\rho_D)\sigma_D^2,\\
\sigma_{y_1}^2 &= \left[2-\frac{2(s_2-s_1s_2-s_2^2)}{(1-s_1)^2}\right](1-\rho_D)\sigma_D^2,\\
\sigma_{y_2}^2 &= \left[2-\frac{2(s_1-s_1s_2-s_1^2)}{(1-s_2)^2}\right](1-\rho_D)\sigma_D^2,\\
\overline{\sigma}'_{y_1}\overline{\sigma}_{y_2} &= \left[\frac{(1+s_1(-3+2s_1)-3s_2+2s_1s_2+2s_2^2)}{(1-s_1)(1-s_2)}\right](1-\rho_D)\,\sigma_D^2.
\end{aligned}$$

Following Bhamra (2007), I use a perturbation expansion of the form:

$$f_1^{\mathcal{A}}=f_{1,0}^{\mathcal{A}}+\epsilon f_{1,1}^{\mathcal{A}}+\epsilon^2 f_{1,2}^{\mathcal{A}}+\dots \tag{A.90}$$

Defining $\rho_D=1-2\epsilon^2$, I get:

$$\begin{aligned}
f_{1,0}^{\mathcal{A}} &= \frac{1}{\delta+e^{-y_1}\delta},\\
f_{1,1}^{\mathcal{A}} &= 0,\\
f_{1,2}^{\mathcal{A}} &= \frac{e^{y_1}\left(1-e^{y_1}\left(-1+s_1\right)^2+s_1^2+2s_1\left(-1+s_2\right)+2\left(-1+s_2\right)s_2\right)\sigma_D^2}{\left(1+e^{y_1}\right)^3\left(-1+s_1\right)^2\delta^2},\\
f_{1,3}^{\mathcal{A}} &= 0.
\end{aligned}$$

After simplification, I obtain:

$$f_1^{\mathcal{A}} = \frac{s_1}{\delta} - \frac{s_1 \left(1 - 3s_1 + 2s_1^2 - 2s_2 + 2s_1 s_2 + 2s_2^2\right)\left(-1 + \rho_D\right)\sigma_D^2}{2\delta^2} + O(\epsilon^4). \tag{A.91}$$

B.9 Solving for $f_{i,t}^{\mathcal{I}}$

Remember that

$$f_{i,t}^{\mathcal{I}} = E_t\left[\int_t^{\infty} e^{-\delta(\tau - t)}\frac{\lambda_\tau}{\lambda_t} s_{i,\tau} d\tau\right], \tag{A.92}$$

which depends on $y_{1,t}$, $y_{2,t}$ and $\nu_{A,t} = \frac{1}{1+\lambda_t}$. Note that λ is a local martingale and that assuming σ_λ is bounded, then it is an exponential martingale. I can then define a new measure:[6]

$$\mathbb{P}'(A_T) = E_t\left[1_{A_T}\lambda_T\right], \quad \forall t, \quad T \in [0, \infty) \quad t \leq T. \tag{A.93}$$

With this change of measure,

$$f_{i,t}^{\mathcal{I}} = E_t^{\mathbb{P}'}\left[\int_t^{\infty} e^{-\delta(\tau - t)} s_{i,\tau} d\tau\right]. \tag{A.94}$$

From (A.94), it follows that $f_{i,t}^{\mathcal{I}}$ satisfies a BSDE. The coefficients of the BSDE will depend on $\nu_{i,t}^{\mathcal{A}}$, which satisfies a FSDE. Together they form a FBSDE. The Feynman-Kac theorem still applies thus $f_{i,t}^{\mathcal{I}}$ satisfies the following inhomogeneous elliptic PDE:

$$\begin{aligned}
&\mu_{y_1}^{\mathbb{P}'}\frac{\partial f_1^{\mathcal{I}}}{\partial y_1} + \mu_{y_2}^{\mathbb{P}'}\frac{\partial f_1^{\mathcal{I}}}{\partial y_2} + \mu_{\nu_{\mathcal{A}}}^{\mathbb{P}'}\frac{\partial f_1^{\mathcal{I}}}{\partial \nu_{\mathcal{A}}} + \frac{1}{2}\sigma_{y_1}^2\frac{\partial^2 f_1^{\mathcal{I}}}{\partial y_1^2} + \frac{1}{2}\sigma_{y_2}^2\frac{\partial^2 f_1^{\mathcal{I}}}{\partial y_2^2} + \frac{1}{2}\sigma_{\nu_{\mathcal{A}}}^2\frac{\partial^2 f_1^{\mathcal{I}}}{\partial \nu_{\mathcal{A}}^2} \\
&\quad + \overline{\sigma}_{y_1}'\overline{\sigma}_{y_2}\frac{\partial^2 f_1^{\mathcal{I}}}{\partial y_1 \partial y_2} + \overline{\sigma}_{y_1}'\overline{\sigma}_{\nu_{\mathcal{A}}}\frac{\partial^2 f_1^{\mathcal{I}}}{\partial y_1 \partial \nu_{\mathcal{A}}} + \overline{\sigma}_{y_2}'\overline{\sigma}_{\nu_{\mathcal{A}}}\frac{\partial^2 f_1^{\mathcal{I}}}{\partial y_2 \partial \nu_{\mathcal{A}}} - \rho f_1^{\mathcal{I}} + \frac{1}{1 + e^{-y_1}} = 0,
\end{aligned} \tag{A.95}$$

[6] See pages 28-29 of Karatzas and Shreve (1998) for details.

where

$$
\begin{aligned}
\mu^{\mathbb{P}'}_{y_1} &= \mu_{y_1} + \overline{\sigma}'_{y_1}\overline{\sigma}_\lambda, \\
\mu^{\mathbb{P}'}_{y_2} &= \mu_{y_2} + \overline{\sigma}'_{y_2}\overline{\sigma}_\lambda, \\
\mu^{\mathbb{P}'}_{\nu_\mathcal{A}} &= \mu_{\nu_\mathcal{A}} + \overline{\sigma}'_{\nu_\mathcal{A}}\overline{\sigma}_\lambda \\
&= -\nu_\mathcal{A}^2(1-\nu_\mathcal{A})\sigma_\lambda^2, \\
\overline{\sigma}'_{y_1}\overline{\sigma}_{\nu_A} &= -\nu_\mathcal{A}(1-\nu_\mathcal{A})\overline{\sigma}'_{y_1}\overline{\sigma}_\lambda, \\
\overline{\sigma}'_{y_2}\overline{\sigma}_{\nu_A} &= -\nu_\mathcal{A}(1-\nu_\mathcal{A})\overline{\sigma}'_{y_2}\overline{\sigma}_\lambda, \\
\overline{\sigma}'_{y_1}\overline{\sigma}_\lambda &= \overline{\sigma}'_{D_1}\overline{\sigma}_\lambda - \left(\frac{s_2}{1-s_1}\right)\overline{\sigma}'_{D_2}\overline{\sigma}_\lambda - \left(1-\frac{s_2}{1-s_1}\right)\overline{\sigma}'_{D_3}\overline{\sigma}_\lambda, \\
\overline{\sigma}'_{y_2}\overline{\sigma}_\lambda &= \overline{\sigma}'_{D_2}\overline{\sigma}_\lambda - \left(\frac{s_1}{1-s_2}\right)\overline{\sigma}'_{D_1}\overline{\sigma}_\lambda - \left(1-\frac{s_1}{1-s_2}\right)\overline{\sigma}'_{D_3}\overline{\sigma}_\lambda.
\end{aligned}
$$

Note that $\sigma^2_{\nu_\mathcal{A}}$ also depends on σ_λ^2 and that $\overline{\sigma}_\lambda$ (and σ_λ^2) depends on the endogenously determined $\overline{\sigma}$.

B.9.1 Boundary conditions

The required boundary conditions are the following:

$$\lim_{y_1\to-\infty} f^\mathcal{I}_{1,t} = 0, \tag{A.96}$$

$$\lim_{y_1\to\infty} f^\mathcal{I}_{1,t} = \frac{1}{\delta}, \tag{A.97}$$

$$\lim_{y_2\to\infty} f^\mathcal{I}_{1,t} = 0, \tag{A.98}$$

$$\lim_{\nu_\mathcal{A}\to 1} \nu_\mathcal{I} f^\mathcal{I}_{1,t} = 0, \tag{A.99}$$

$$\left.\frac{\partial f^\mathcal{I}_{1,t}}{\partial \nu_\mathcal{A}}\right|_{\nu_\mathcal{A}=0} = 0. \tag{A.100}$$

Finally, when $y_2 \to -\infty$, then the second dividend tree becomes irrelevant and $f^{\mathcal{I}}_{1,t}$ converges to the case of Bhamra (2007). The other boundary conditions are justified as follows:

1. $\lim_{y_1 \to -\infty} f^{\mathcal{I}}_{1,t} = 0$ and $\lim_{y_2 \to \infty} f^{\mathcal{I}}_{1,t} = 0$: When $y_2 \to \infty$, I must be that $y_1 \to -\infty$. When $y_1 \to -\infty$, the first dividend stream becomes irrelevant so investors aren't willing to pay anything to own the stock.

2. $\lim_{y_1 \to \infty} f^{\mathcal{I}}_{1,t} = \frac{1}{\delta}$: In this case there is a single dividend tree and complete markets (the constraint becomes irrelevant), so:

$$\begin{aligned} S_1 &= \frac{D_1}{\delta} = D_M(\nu_{\mathcal{A},t} f^{\mathcal{A}}_{1,t} + \nu_{\mathcal{I},t} f^{\mathcal{I}}_{1,t}), \\ \Rightarrow \frac{1}{\delta} &= \nu_{\mathcal{A},t}\left(\frac{1}{\delta}\right) + (1 - \nu_{\mathcal{A},t}) f^{\mathcal{I}}_{1,t} = f^{\mathcal{I}}_{1,t}. \end{aligned}$$

3. $\lim_{\nu_{\mathcal{A}} \to 1} \nu_{\mathcal{I}} f^{\mathcal{I}}_{1,t} = 0$: When $\nu_{\mathcal{A}} = 1$, agent $\mathcal{A}$, which faces no constraint, consumes all dividends so markets are complete. Therefore $f_{1,t}\Big|_{\nu_{\mathcal{A}}=1} = f^{\mathcal{A}}_{1,t}$ so this boundary condition must hold.

4. $\frac{\partial f^{\mathcal{I}}_{1,t}}{\partial \nu_{\mathcal{A}}}\Bigg|_{\nu_{\mathcal{A}}=0} = 0$: As $\nu_{\mathcal{A}} \to 0$, indexers consume all dividends. However, they have a worst investment opportunity set than active investors, so this can't hold for more than an instant. $f^{\mathcal{I}}_{1,t}$ is also bounded by $1/\delta$, which occurs when indexers consume all dividends. Because $f^{\mathcal{I}}_{1,t}$ cannot grow outside the domain of $\nu_{\mathcal{A}} = 0$, the Neumann condition must be a reflecting boundary condition where the derivate is equal to 0. Therefore this boundary condition must be a reflecting boundary condition.

B.10 Matching moments

I now have expressions for both $f_{i,t}^{\mathcal{A}}$ and $f_{i,t}^{\mathcal{I}}$. I have a closed form expression for $f_{i,t}^{\mathcal{A}}$ that depends on exogenous parameters and state variables, which is easy to evaluate numerically. For $f_{i,t}^{\mathcal{I}}$, I have a PDE that can be approximated. However, the current form of that solution depends on the endogenously determined $\overline{\sigma}$ because of the dependence on $\overline{\sigma}_\lambda$. I have that $S_{i,t} = D_{M,t} f_{i,t}$, so:

$$\begin{aligned} dS_i &= D_M df_i + f_i dD_M + df_i dD_M, \\ \frac{dS_i}{S_i} &= \frac{df_i}{f_i} + \frac{dD_M}{D_M} + \frac{df_i}{f_i}\frac{dD_M}{D_M}, \end{aligned} \tag{A.101}$$

where

$$\frac{dD_M}{D_M} = \mu_D dt + \overline{\sigma}_D' d\overline{Z},$$

and $\overline{\sigma}_D = s_1\overline{\sigma}_{D_1} + s_2\overline{\sigma}_{D_2} + (1 - s_1 - s_2)\overline{\sigma}_{D_3}$. I know that $f_{i,t}$ is a function of exogenous parameters and state processes s_1, s_2 and $\nu_{\mathcal{A}}$, therefore from Itô's Lemma I get:

$$\begin{aligned} df_i = &\left[\mu_{\nu_{\mathcal{A}}}\frac{\partial f_i}{\partial \nu_{\mathcal{A}}} + \mu_{s_1}\frac{\partial f_i}{\partial s_1} + \mu_{s_2}\frac{\partial f_i}{\partial s_2} + \frac{1}{2}\left(\sigma_{\nu_{\mathcal{A}}}^2\frac{\partial^2 f_i}{\partial \nu_{\mathcal{A}}^2} + \sigma_{s_1}^2\frac{\partial^2 f_i}{\partial s_1^2} + \sigma_{s_2}^2\frac{\partial^2 f_i}{\partial s_2^2}\right.\right. \\ &\left.\left. + 2\overline{\sigma}_{\nu_{\mathcal{A}}}'\overline{\sigma}_{s_1}\frac{\partial^2 f_i}{\partial \nu_{\mathcal{A}}\partial s_1} + 2\overline{\sigma}_{\nu_{\mathcal{A}}}'\overline{\sigma}_{s_2}\frac{\partial^2 f_i}{\partial \nu_{\mathcal{A}}\partial s_2} + 2\overline{\sigma}_{s_1}'\overline{\sigma}_{s_2}\frac{\partial^2 f_i}{\partial s_1\partial s_2}\right)\right] dt \\ &+ \left[\overline{\sigma}_{\nu_{\mathcal{A}}}\frac{\partial f_i}{\partial \nu_{\mathcal{A}}} + \overline{\sigma}_{s_1}\frac{\partial f_i}{\partial s_1} + \overline{\sigma}_{s_2}\frac{\partial f_i}{\partial s_2}\right]' d\overline{Z}. \end{aligned} \tag{A.102}$$

From the definition of stock return process, I also have that:

$$\frac{dS_i}{S_i} = \left[\mu_i - \frac{D_i}{S_i}\right] dt + \overline{\sigma}_i' d\overline{Z}, \tag{A.103}$$

where

$$
\begin{aligned}
\mu_i = \frac{1}{f_i} \Bigg[& \mu_{\nu_{\mathcal{A}}} \frac{\partial f_i}{\partial \nu_{\mathcal{A}}} + \mu_{s_1} \frac{\partial f_i}{\partial s_1} + \mu_{s_2} \frac{\partial f_i}{\partial s_2} + \frac{1}{2} \left(\sigma_{\nu_{\mathcal{A}}}^2 \frac{\partial^2 f_i}{\partial \nu_{\mathcal{A}}^2} + \sigma_{s_1}^2 \frac{\partial^2 f_i}{\partial s_1^2} + \sigma_{s_2}^2 \frac{\partial^2 f_i}{\partial s_2^2} \right. \\
& \left. + 2\overline{\sigma}_{\nu_{\mathcal{A}}}' \overline{\sigma}_{s_1} \frac{\partial^2 f_i}{\partial \nu_{\mathcal{A}} \partial s_1} + 2\overline{\sigma}_{\nu_{\mathcal{A}}}' \overline{\sigma}_{s_2} \frac{\partial^2 f_i}{\partial \nu_{\mathcal{A}} \partial s_2} + 2\overline{\sigma}_{s_1}' \overline{\sigma}_{s_2} \frac{\partial^2 f_i}{\partial s_1 \partial s_2} \right) \\
& + \left(\overline{\sigma}_{\nu_{\mathcal{A}}} \frac{\partial f_i}{\partial \nu_{\mathcal{A}}} + \overline{\sigma}_{s_1} \frac{\partial f_i}{\partial s_1} + \overline{\sigma}_{s_2} \frac{\partial f_i}{\partial s_2} \right)' \overline{\sigma}_D \Bigg] + \mu_D, \qquad \text{(A.104)}
\end{aligned}
$$

$$
\overline{\sigma}_i = \frac{1}{f_i} \left[\overline{\sigma}_{\nu_{\mathcal{A}}} \frac{\partial f_i}{\partial \nu_{\mathcal{A}}} + \overline{\sigma}_{s_1} \frac{\partial f_i}{\partial s_1} + \overline{\sigma}_{s_2} \frac{\partial f_i}{\partial s_2} \right] + \overline{\sigma}_D. \qquad \text{(A.105)}
$$

Note that the expression I have for $\overline{\sigma}_\lambda$ from (A.44) is a function of both $\overline{\sigma}$ and the equilibrium price ratio f_1/f_2, since $\omega_1^I = 1 + f_1/f_2$ and $\omega_2^I = 1 + f_2/f_1$. I first use the definitions of $\overline{\sigma}$ and $\overline{\sigma}_\lambda$ to create perturbation expansions of these moments as a function of f_1, f_2, f_3 and their own expansions. Substituting these expansions in the PDE (A.95), I create a perturbation expansion of the PDE, and then solve by equating terms in the different powers of ϵ. The result is the closed-form approximation

$$
\begin{aligned}
f_1^{\mathcal{I}} = f_1^{\mathcal{A}} + & \frac{1}{2(s_1 + s_2)\nu_{\mathcal{A}}\delta^2} s_1 \Big(2(-1 + s_1 + s_2) \Big(-s_2 + 2 \left(s_1^2 + s_1(-1 + s_2) + s_2^2 \right) \Big) \\
& + (s_1 + s_2) \Big(1 + 2s_1^2 + 2(-1 + s_2)s_2 + s_1(-3 + 2s_2) \Big) \nu_{\mathcal{A}} \Big) (1 - \rho_D) \sigma_D^2 + O(\epsilon^4).
\end{aligned}
\qquad \text{(A.106)}
$$

As in the unconstrained economy, I find $f_2^{\mathcal{I}}$ by symmetry and $f_3^{\mathcal{A}}$ by $f_3^{\mathcal{I}} = \frac{1}{\delta} - f_1^{\mathcal{I}} - f_2^{\mathcal{I}}$. A drawback of the use of a perturbation expansion is that it is impossible to guarantee that the boundary conditions will be satisfied. It is easy to see that in this case (A.100) is not satisfied, which means that the approximation will not be valid in the neighborhood of $\nu_{\mathcal{A}} = 0$. Since this region is not economically important for the current analysis,[7] this does not pose a problem as long as the analysis focuses on values of $\nu_{\mathcal{A}}$ that are away from that boundary.

C Vector notation

This section introduces the two different vector bases I use in the proofs. While not a necessary read, this section is a useful appendix for understanding the proofs. The reason for using different bases is to simplify certain steps of the proof. Steps involving stock returns are easier to solve under the market basis. However, when solving for equilibrium stock return dynamics, the dividend basis is more appropriate. The dividend processes in (A.1) can be represented as a vector:

$$\frac{dD_t}{D_t} = \mu_D \mathbf{1} dt + \sigma_D \mathbf{1}' dZ_{D_t}, \tag{A.1}$$

where $\frac{dD_t}{D_t}$ is a vector with $\frac{dD_{i,t}}{D_{i,t}}$ as the i-th element and dZ_{D_t} is a vector with $dZ_{D_{i,t}}$ as the i-th element. Since the $dZ_{D_{i,t}}$ can be correlated, we can represent the correlation matrix of dZ_{D_t} as

$$C_{D_t} = \begin{bmatrix} 1 & \rho_D & \rho_D \\ \rho_D & 1 & \rho_D \\ \rho_D & \rho_D & 1 \end{bmatrix}.$$

Stock returns in (A.4) can also be represented in vector notation:

$$dR_t = \mu_t dt + \sigma_t dZ_t,$$

[7] $\nu_{\mathcal{A}} = 0$ corresponds to the case where the aggregate wealth is fully owned by the indexer, and the remaining active investor still has to hold the share of the non-index stock. The realization of such a scenario seems highly unlikely.

where dR_t, μ_t and dZ_t are vectors with $dR_{i,t}$, $\mu_{i,t}$ and $dZ_{i,t}$ as the i-th element and σ_t is a diagonal matrix with $\sigma_{i,t}$ as the i-th diagonal element. The dZ_t BM are correlated with correlation matrix:

$$C_t = \begin{bmatrix} 1 & \rho_{t,12} & \rho_{t,13} \\ \rho_{t,12} & 1 & \rho_{t,23} \\ \rho_{t,13} & \rho_{t,23} & 1 \end{bmatrix}.$$

C.1 Rotation matrix

It is often easier to deal with independent Brownian motions (BM) than correlated ones. It is possible to transform a multivariate BM to a vector of independent BM using a rotation matrix. Under that transformation, drifts, variances and covariances of Itô processes are invariant. Consider the three-dimensional multivariate BM $Z = [Z_1\ Z_2\ Z_3]'$ with correlation matrix:

$$C = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{bmatrix}.$$

Using the Cholesky decomposition, we can construct a rotation matrix K to transform Z into a three-dimensional vector of independent BM. From the Cholesky decomposition, we get the lower triangular matrix L such that $LL' = C$. The matrix L is often used to generate correlated BM from independent ones such that $Z = LX$. In this case, I am interested in the inverse process: $X = KZ$ where $K = L^{-1}$.

Applying the Cholesky decomposition to the matrix C,

$$K = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{\rho_{12}}{\sqrt{1-\rho_{12}^2}} & \frac{1}{\sqrt{1-\rho_{12}^2}} & 0 \\ \frac{\rho_{13}-\rho_{12}\rho_{23}}{\sqrt{(1-\rho_{12}^2)(1-\rho_{12}^2-\rho_{13}^2+2\rho_{12}\rho_{13}\rho_{23}-\rho_{23}^2)}} & \frac{-\rho_{12}\rho_{13}+\rho_{23}}{\sqrt{(1-\rho_{12}^2)(1-\rho_{12}^2-\rho_{13}^2+2\rho_{12}\rho_{13}\rho_{23}-\rho_{23}^2)}} & \frac{1}{\sqrt{1+\frac{\rho_{13}^2-2\rho_{12}\rho_{13}\rho_{23}+\rho_{23}^2}{-1+\rho_{12}^2}}} \end{bmatrix}. \tag{A.2}$$

Changing the set of BMs using a rotation matrix is called a change of basis. Drift terms, total variances and covariances between processes are invariant under a change of basis. Note that if the initial BM are uncorrelated (correlation terms in C all equal to 0), then the rotation matrices L and K collapse to the identity matrix.

C.2 Dividend basis

The BM driving the dividend processes described in (A.1) are correlated. Consider L_{D_t}, the lower triangular matrix from the Cholesky decomposition of C_{D_t}, and it's inverse K_{D_t}. Then I can rewrite (A.1) as:

$$\begin{aligned} \frac{dD_t}{D_t} &= \mu_D dt + \sigma_D dZ_{D_t} \\ &= \mu_D dt + \sigma_D L_D \overline{Z}_{D_t} \\ &= \mu_D dt + \overline{\sigma}_D \overline{Z}_{D_t}, \end{aligned}$$

where $\overline{\sigma}_D = \sigma_D L_D$ and $\overline{Z}_{D_t} = K_D Z_{D_t}$. This transformation yields a new basis that I call the dividend basis. The variance matrix under the dividend basis can be written as:

$$\overline{\sigma}_D = \begin{pmatrix} 1 & 0 & 0 \\ \rho_D & \sqrt{1-\rho_D^2} & 0 \\ \rho_D & \frac{\sqrt{1-\rho_D}\rho_D}{\sqrt{1+\rho_D}} & \sqrt{3-2\rho_D-\frac{2}{1+\rho_D}} \end{pmatrix}. \tag{A.3}$$

C.3 Market basis

Similarly, the BM driving the market return processes in (A.4) might be correlated as they are determined endogenously. Consider L_t, the lower triangular matrix from the

Cholesky decomposition of C_t, and it's inverse K_t. Then I can write:

$$\begin{aligned} dR_t &= \mu_t dt + \sigma_t dZ_t \\ &= \mu_t dt + \sigma_t L_t d\underline{Z}_t \\ &= \mu_t dt + \underline{\sigma}_t d\underline{Z}_t, \end{aligned}$$

where $\underline{\sigma}_t = \sigma_t L_t$ and $\underline{Z}_t = K_t Z_t$. This transformation yields a new basis that I call the market basis. Under this basis,

$$\underline{\sigma} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ \rho_{12}\sigma_2 & \sqrt{1-\rho_{12}^2}\sigma_2 & 0 \\ \rho_{13}\sigma_3 & \frac{-\rho_{12}\rho_{13}+\rho_{23}}{\sqrt{1-\rho_{12}^2}}\sigma_3 & \sqrt{1+\frac{\rho_{13}^2-2\rho_{12}\rho_{13}\rho_{23}+\rho_{23}^2}{-1+\rho_{12}^2}}\sigma_3 \end{bmatrix}. \tag{A.4}$$

Note that the return process can also be written under the dividend basis as:

$$dR_t = \mu_t dt + \overline{\sigma}_t d\overline{Z}_{D_t},$$

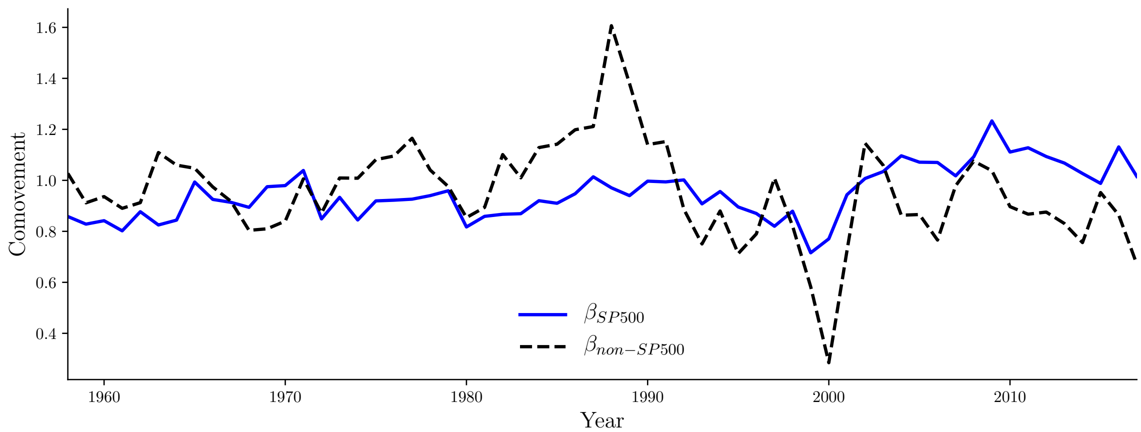
where $\overline{\sigma}_t d\overline{Z}_{D_t} = \sigma_t dZ_t = \underline{\sigma}_t d\underline{Z}_t$. $\overline{\sigma}_t$ has the generic form:

$$\overline{\sigma}_t = \begin{bmatrix} \overline{\sigma}_{11} & \overline{\sigma}_{12} & \overline{\sigma}_{13} \\ \overline{\sigma}_{21} & \overline{\sigma}_{22} & \overline{\sigma}_{23} \\ \overline{\sigma}_{31} & \overline{\sigma}_{32} & \overline{\sigma}_{33} \end{bmatrix}. \tag{A.5}$$

However, this leaves 9 unknowns to solve for in $\overline{\sigma}_t$ (it is a 3×3 matrix), whereas the known structure of $\underline{\sigma}_t$ leaves only 6 unknowns to solve for, namely σ_1, σ_2, σ_3, ρ_{12}, ρ_{13} and ρ_{23}.

D Figures and Tables

Figure A.1. Average Comovement of S&P 500 Stocks from 1957 to 2017 - Univariate Regressions



This figure presents the average annual comovement βs of S&P 500 stocks from 1957 to 2017 estimated from the univariate regressions:

$$R_{j,t} = \alpha_{SP500,j} + \beta^{univ}_{j,\text{SP500},t} R_{\text{SP500},t} + u_{SP500,j,t}$$
$$R_{j,t} = \alpha_{nonSP500,j} + \beta^{univ}_{j,\text{nonSP500},t} R_{\text{nonSP500},t} + u_{nonSP500,j,t},$$

The sample includes stocks in the index on the last trading day of the year, using returns from the following 12 months.

Table A.1. Changes in Comovement on Lagged Changes in Passive Ownership - Univariate Regressions

This table presents regression estimates of changes in average comovement for S&P 500 firms ($\Delta\beta^{univ}_{\text{SP500},t}$ and $\Delta\beta^{univ}_{\text{nonSP500},t}$) on lagged changes in passive ownership (ΔPO) and on $\Delta\beta^{univ}_{t-1}$, the lagged value of the dependent variable. The dependent variables are the changes in average β^{univ}s. Comovement β^{univ}s are estimated from the regressions

$$R_{j,t} = \alpha_{SP500,j} + \beta^{univ}_{j,\text{SP500},t} R_{\text{SP500},t} + u_{SP500,j,t} \quad R_{j,t} = \alpha_{nonSP500,j} + \beta^{univ}_{j,\text{nonSP500},t} R_{\text{nonSP500},t} + u_{nonSP500,j,t},$$

where $R_{\text{SP500},t}$ is the value-weighted return of the S&P 500 stocks portfolio (excluding stock j) and $R_{\text{non-SP500},t}$ is the value-weighted return of the rest of the market. Comovement βs are estimated based on index membership at the end of December, using daily returns for the following 12 months. PO is total passive assets under management divided by the total CRSP market capitalization and acts as a proxy for the relative wealth of indexers. Newey-West standard errors based on three lags and standard errors based on a moving block bootstrap with three lags are presented in parenthesis and square brackets, respectively, and bootstrapped p-values in italics. The bootstrapped distribution is based on 100,000 samples. The sample period is 1985 to 2017.

	$\Delta\beta_{SP500,t}$			$\Delta\beta_{nonSP500,t}$		
	(1)	(2)	(3)	(4)	(5)	(6)
Intercept	0.006	0.005	0.007	0.001	-0.014	0.002
	(0.01)	(0.01)	(0.01)	(0.03)	(0.03)	(0.03)
	0.64	*0.68*	*0.62*	*0.96*	*0.60*	*0.95*
ΔPO_{t-1}	-1.735		-1.671	-9.924		-9.872
	(2.30)		(2.79)	(8.86)		(8.61)
	0.46		*0.55*	*0.27*		*0.26*
$\Delta\beta_{t-1}$		-0.227	-0.225		0.053	0.047
		(0.17)	(0.16)		(0.13)	(0.10)
		0.19	*0.17*		*0.68*	*0.65*
Adj. R^2	-0.021	0.016	-0.007	0.026	-0.030	-0.005
N	32	32	32	32	32	32

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